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Corollary and Application of Argument Principle

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Abstract

From the point of view of the properties of argument principle, this paper uses the residue theorem and residue operation to get some important inferences of argument principle.First, C is a circumferential line, and function $f(z)$、 $\varphi(z)$ satisfies the condition that $f(z)$ is meromorphic inside C. $\varphi(z)$ resolves on the closed field $\overline{I(C)}$ and $f(z)$ resolves on C without zeros. So, $f(z)$ has different zeros and poles inside the perimeter of C that satisfy an integral equation. Second, let C be a circumferential line, $\forall a \in R$, and $f(z)-a$ satisfy that it is meromorphic inside C. It resolves on C and has no zeros.So, there's an expression for a logarithmic residue of $f(z)-a$ .By analyzing typical problems, this paper discusses the argument principle and corollary in the complex field, including the number of zeros and distribution of the polynomial (or rational function) in a given region.

Keywords

Argument Principle; Rouché Theorem; Logarithmic Residues; Meromorphic Functions.

1. Introduction

In complex function theory, Cauchy integral theorem, Cauchy integral formula and residue theorem can be used to solve the perimeter integral. The uniqueness theorem examines the distribution of zeros of an analytic function in a given region. However, it is usually difficult to determine the number of zeros. In complex domain, the argument principle can solve some perimeter integrals and determine the number and distribution of zeros of functions in a given region. In addition, it has a good application in the judgment of Nai's stability in automatic control theory. There is still room for extension of the theory of argument. This paper discusses the principle, corollary and application.

2. Prepare Knowledge

Definition 1. If function $\frac{f'(z)}{f(z)}=\frac{d}{dz}[\ln f(z)]$, the integral $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$ is said to be the logarithmic residue of $f(z)$ with respect to perimeter C.

Definition 2[1]. A single-valued analytic function that has no singularities of any kind on the z-plane other than poles is called meromorphic.

Lemma 1[1] (residue theorem). Let C be a perimeter and function $f(z)$ satisfy the following conditions. (1) $f(z)$ resolves on C. (2) $f(z)$ resolves inside C except for n isolated singularities $a_1,a_2,\dots,a_k$, then there is $\int_C f(z)dz = 2\pi i\cdot\sum_{k=1}^{n}\operatorname*{Res}_{z=a_k} f(z)$.

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Lemma 2[1]. (1) Let a be the nth-order zero of function $f(z)$, then A must be the first-order pole of function $\frac{f'(z)}{f(z)}$, and have $\operatorname{Res}_{z=a}\left[\frac{f'(z)}{f(z)}\right]=n$. (2) Let b be the m-order pole of function $f(z)$, then b must be the first-order pole of function $\frac{f'(z)}{f(z)}$, and $\operatorname{Res}_{z=b}\left[\frac{f'(z)}{f(z)}\right]=-m$ is obtained.

Lemma 3[1] (argument principle). Let C be a circumferential line and function $f(z)$ satisfy the following conditions. (1) $f(z)$ is meromorphic in C, (2) $f(z)$ resolves on C and has no zero, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N(f,C)-P(f,C)=\frac{\Delta_C \arg f(z)}{2\pi}$.

Where, $N(f,C)$ and $P(f,C)$ respectively represent the number of zeros and poles of function C inside the perimeter line $f(z)$ (an n-order zero is denoted as n zeros, and an m-order pole as m poles). $\Delta_C \arg f(z)$ represents the sum of $\arg f(z)$ as z travels around C in the positive direction is an integer multiple of 2π. In particular, if the function $f(z)$ resolves on C and the interior of C and has no zero on C, there is $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N(f,C)=\frac{\Delta_C \arg f(z)}{2\pi}$.

Note: If the condition " $f(z)$ resolves on perimeter C and has no zeros" is reduced to " $f(z)$ continues to the boundary of perimeter C and has no zeros", the conclusion of the argument principle still holds.

Lemma 4[2]. Let the polynomial $P(z)=a_0+a_1z+\cdots+a_tz^t+\cdots+a_nz^n$ have no zeros on the imaginary axis. If z goes from ∞ to ∞ along the imaginary axis from top to bottom, $P(z)$ goes k times around the origin, and has $\Delta\arg_{y(-\infty\to+\infty)} P(iy)=2k\pi$. So $P(z)$ has $\frac{n+2k}{2}$ zeros in $\operatorname{Re} z<0$ on the left half plane.

3. Corollary of Argument Principle

Lemma $2'$. (1) Let a_k be the n_k-order zero of function $f(z)$, $\varphi(z)$ resolved at point a_k. So a_k must be the first pole of $\varphi(z)\frac{f'(z)}{f(z)}$, and have $\operatorname{Res}_{z=a_k}\left[\varphi(z)\frac{f'(z)}{f(z)}\right]=n_k\varphi(a_k)$. (2) If b_j is set as the m_j-order pole of function $f(z)$, b_j must be the first-order pole of function $\varphi(z)\frac{f'(z)}{f(z)}$, and have $\operatorname{Res}_{z=b_j}\left[\varphi(z)\frac{f'(z)}{f(z)}\right]=-m_j\varphi(b_j)$.

Proof. (1) Since a_k is the n_k-order zero of function $f(z)$, there is a_k in the neighborhood of point $f(z)=(z-a_k)^{n_k}g(z)$. Where $g(z)$ resolves in the neighborhood of point a_k and $g(a_k)\neq 0$. So, there are $f'(z)=n_k(z-a_k)^{n_k-1}g(z)+(z-a_k)^{n_k}g'(z)$. Since $\varphi(z)\cdot\frac{g'(z)}{g(z)}$ resolves in the neighborhood of point a_k, a_k must be the first-order pole of $\varphi(z)\frac{f'(z)}{f(z)}$ and $\operatorname{Res}_{z=a_k}\left[\varphi(z)\frac{f'(z)}{f(z)}\right]=n_k\varphi(a_k)$. (2)

Since b_j is the m_j pole of function $f(z)$, there is $f(z)=\frac{h(z)}{(z-b_j)^{m_j}}$ in the centroid neighborhood of b_j. Where $h(z)$ resolves in the neighborhood of point b_j and $h(b_j)\neq 0$. So, there are $\varphi(z)\frac{f'(z)}{f(z)}=\varphi(z)\cdot\frac{-m_j}{z-b_j}+\varphi(z)\cdot\frac{h'(z)}{h(z)}$. Since $\varphi(z)\cdot\frac{h'(z)}{h(z)}$ resolves in the neighborhood of b_j, b_j must be the first pole of $\varphi(z)\frac{f'(z)}{f(z)}$ and $\operatorname*{Res}_{z=b_j}\left[\varphi(z)\frac{f'(z)}{f(z)}\right]=-m_j\varphi(b_j)$. End of proof.

Note . (1) when $\varphi(a_k)=0$, function $\varphi(z)\frac{f'(z)}{f(z)}$ resolves at point a_k, $\operatorname*{Res}_{z=a_k}\left[\varphi(z)\frac{f'(z)}{f(z)}\right]=0$.(2) Either $\varphi(b_j)\neq 0$ or $\varphi(b_j)=0$, $\operatorname*{Res}_{z=b_j}\left[\varphi(z)\frac{f'(z)}{f(z)}\right]=-m_j\varphi(b_j)$.

Theorem 1. Let C be a perimeter and function $f(z)$、$\varphi(z)$ satisfy the following condition. (1) $f(z)$ is meromorphic inside C, $\varphi(z)$ is resolved on closed field $\overline{I(C)}$, (2) $f(z)$ is resolved on C and has no zeros, so $\frac{1}{2\pi i}\int_C\varphi(z)\frac{f'(z)}{f(z)}dz=\sum_{k=1}^{p}n_k\varphi(a_k)-\sum_{j=1}^{q}m_j\varphi(b_j)$.

Where, $a_k(k=1,2,...,p)$ means that function $f(z)$ has different zeros and order n_k inside the perimeter C. $b_j(j=1,2,...,q)$ indicates that function $f(z)$ has different poles and order m_j inside perimeter C.

Proof. from lemma 1 and lemma, it can be obtained $\frac{1}{2\pi i}\int_C\varphi(z)\frac{f'(z)}{f(z)}dz=\sum_{k=1}^{p}\operatorname*{Res}_{z=a_k}\left[\varphi(z)\frac{f'(z)}{f(z)}\right]+\sum_{j=1}^{q}\operatorname*{Res}_{z=b_j}\left[\varphi(z)\frac{f'(z)}{f(z)}\right]==\sum_{k=1}^{p}n_k\varphi(a_k)-\sum_{j=1}^{q}m_j\varphi(b_j)$, End of proof.

Note. when $\varphi(z)=1$, it is lemma 3.

Theorem 2. Let C be a perimeter, and for $\forall a\in R$, function $f(z)-a$ satisfies the following condition. (1) $f(z)-a$ is meromorphic inside C, (2) $f(z)-a$ resolves on C and has no zeros, then $\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)-a}dz=N(f-a,C)-P(f-a,C)=\frac{\Delta_C\arg[f(z)-a]}{2\pi}$.

Proof. From the condition, the function $f(z)-a$ has at most a finite number of zeros and poles inside C. For $\forall a\in R$, function $f(z)-a$ also satisfies the argument principle. End of proof.

In particular, if the function $f(z)-a$ resolves on C and the interior of C and has no zero on C, there is $\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)-a}dz=N(f-a,C)=\frac{\Delta_C\arg[f(z)-a]}{2\pi}$.

Theorem 3[3]. Let C be a perimeter and function $f(z)$、$\varphi(z)$ satisfy the following conditions. (1) $f(z)$、$\varphi(z)$ is meromorphic in C, (2) $f(z)$、$\varphi(z)$ resolves on C and has no zeros, and (3) $L=\left\{\omega(z)=\frac{\varphi(z)}{f(z)}\middle| z\in C\right\}$ satisfies the undivided 0 and ∞ points in C, then $N(f,C)-P(f,C)=N(\varphi,C)-P(\varphi,C)$.

Theorem 4[4] (Rouché theorem). Let C be a circumferential line and functions $f(z)$ and $\varphi(z)$ satisfy the following conditions. (1) They are all resolved inside C and continue to C; (2) On C,

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$|f(z)| > |\varphi(z)|$, then functions $f(z)$ and $f(z)+\varphi(z)$ have the same number of zeros inside C, that is, $N(f+\varphi,C)=N(f,C)$.

4. Application of Argument Principle

4.1 Calculate Some Perimeter Integrals

In the complex domain, compute the perimeter integral. We need to consider the analyticity of the integrand and whether there are singularities in the peripheral region. Then, the Cauchy integral theory is used to solve the problem. In addition, the argument principle is also efficient in solving the perimeter integral.

Proposition 1. Evaluate the integral $\int_{|z|=1} \frac{1}{z^2+2z} dz$.

Solution. Original formula $= \int_{|z|=1} \frac{1}{z(z+2)} dz$, For $\int_{|z|=1} \frac{1}{z} dz$, $\int_{|z|=1} \frac{1}{z+2} dz$, since $f(z)=z$, $g(z)=z+2$ resolves both in C, and on C, $f(z)\neq 0$, $g(z)\neq 0$ has a zero in C and no poles. $g(z)$ has neither zero nor pole in C, so Original formula $=\frac{1}{2}\left(\int_{|z|=1} \frac{1}{z} dz - \int_{|z|=1} \frac{1}{z+2} dz\right) = \frac{1}{2}[2\pi i(1-0)-0]=\pi i$.

This is consistent with the result of Cauchy integral theorem.

4.2 Determine the Number and Distribution of Zeros of Functions in the Specified Area

In some practical problems, it is necessary to know the number of zeros and distribution of certain functions (mainly polynomial or rational functions) in a given region. The argument principle established in residue theory can solve this kind of problem easily.

Proposition 2. If function $f(z)$ resolves on C except for a first-order pole inside the perimeter C, and resolves on C and $|f(z)|=1$, then function $f(z)-a$ $(\forall a \in R, |a|>1)$ has exactly one root in C.

Proof. since $f(z)-a$ satisfies theorem 2, then $N(f-a,C)-P(f-a,C)=\frac{\Delta_C \arg[f(z)-a]}{2\pi}$. As z goes around C, $\eta = f(z)$ goes around the circle $\Gamma: |\eta| = |f(z)| = 1$. a $(\forall a \in R, |a|>1)$ is on the outside of Γ, $f(z)-a$ doesn't go around $\eta=0$, $\Delta_C \arg[f(z)-a]=0$, that is, $N(f-a,C)-P(f-a,C)=0$. And since $f(z)$ has a meromorphic function with only one pole inside perimeter C, resolved on C and $|f(z)|=1\neq 0$, which satisfies the argument principle, has $N(f-a,C)=P(f-a,C)=P(f,C)=1$, that is, function $(\forall a \in R, |a|>1)$ has exactly one root in C.

Proposition 3. Let C be a circumferential line, function $f(z)$ is meromorphic in C and continues to C, (1) If $z\in C$, $|f(z)|<1$, it gets $N(f-1,C)=P(f,C)$; (2) If $z\in C$, $|f(z)|>1$, it gets $N(f-1,C)=N(f,C)$.

Proof. $f(z)$ satisfies the argument principle condition and has $N(f,C)-P(f,C)=\frac{\Delta_C \arg f(z)}{2\pi}$.

So $f(z)-1$ satisfies theorem 2, $N(f-1,C)-P(f-1,C)=\frac{\Delta_C \arg[f(z)-1]}{2\pi}$.Therefore, (1) If $z\in C$, $|f(z)|<1$. Because when Z goes around C in the direction of C, $\eta = f(z)$ turns the circle C in the z plane into a closed curve Γ in the η plane, and 1 is outside Γ, that is, $f(z)-1$ does not go around $\eta=0$. If there is $\Delta_C \arg[f(z)-1]=0$, there is $N(f-1,C)=P(f-1,C)=P(f,C)$. End of proof. (2)

Because $f(z)-1=f(z)\cdot\left[1-\frac{1}{f(z)}\right]$, $\Delta_C \arg[f(z)-1]=\Delta_C \arg f(z)+\Delta_C \arg\left(1-\frac{1}{f(z)}\right)$. Let $\eta=1-\frac{1}{f(z)}$, because $\eta=1-\frac{1}{f(z)}$ turns the circle C in the z plane into a closed curve Γ in the η plane as z goes around C in the forward direction. Since $|f(z)|>1$, Γ is contained inside the circle $|\eta-1|=1$, and η does not go around the origin $\eta=0$, $\Delta_C \arg\eta=\Delta_C \arg\left[1-\frac{1}{f(z)}\right]=0$, $\Delta_C \arg[f(z)-1]=\Delta_C \arg f(z)$. So $N(f-1,C)-P(f-1,C)=N(f,C)-P(f,C)$, $P(f-1,C)=P(f,C)$, then $N(f-1,C)=N(f,C)$. End of proof.

Proposition 4. Find how many roots does the polynomial $P(z)=z^7-6z^5+z^2-3$ have in $|z|<1$.

Solution. take $f(z)=-6z^5$, $\varphi(z)=z^7+z^2-3$, on $|z|=1$, $|f(z)|=|-6z^5|=6$, $|\varphi(z)|=|z^7+z^2-3|=1$, that $|f(z)|>|\varphi(z)|$ satisfied the condition of Rouché theorem, $f(z)+\varphi(z)$ and $f(z)$ have the same number of roots at $|z|<1$, i.e. $N(z^7-6z^5+z^2-3,|z|=1)=N(-6z^5,|z|=1)=6$. Thus, the polynomial $P(z)=z^7-6z^5+z^2-3$ has six roots in $|z|<1$.

Proposition 5. Test. The polynomial $P(z)=z^5+z^2+1$ has three zeros in the left half plane.

Analysis. According to lemma 4, we only need to prove that the following two points are true. (1) $P(z)$ has no zero on the imaginary axis; (2) $\underset{y(-\infty\to+\infty)}{\Delta\arg} P(z)=\pi$.

Proof. (1) on the imaginary axis: $z=iy$, $(-\infty<y<+\infty)$ has $P(iy)=(iy)^5+(iy)^2+1=(1-y^2)+iy^5$. So when $y=\pm1$, $\operatorname{Re}P(iy)==1-y^2=0$. $\operatorname{Im}P(iy)=y^5\neq0$, so $P(z)$ has no zero on the imaginary axis. (2) When point z moves from point ∞ to point ∞ along the virtual axis from bottom to top, the trajectory of $\eta=P(iy)=u(y)+iv(y)$ is a curve, and the equation is $\begin{cases}u(y)=1-y^2\\ v(y)=y^5\end{cases}$,$(-\infty<y<+\infty)$, (see figure 1.) When y goes from $-\infty\to-1$, we can see from $v(y)$ that $v(y)$ goes from $-\infty\to-1$. The expression for $u(y)$ shows that $u(y)$ goes from $-\infty\to0$. This point is in the bottom left plane and call it l_1. When y goes from $-1\to0$, we can see from $v(y)$ that $v(y)$ goes from $-1\to0$. The expression for $u(y)$ tells us that $u(y)$ goes from $0\to1$. This point is in the bottom right plane and call it l_2. When y goes from $0\to1$, we can see from $v(y)$ that $v(y)$ goes from $0\to1$.The expression $u(y)$ for $u(y)$ can be seen from $1\to0$. This point is in the upper right plane and call it l_3. When y equals $1\to+\infty$, $v(y)$ equals $v(y)$ from $1\to+\infty$. The expression for $u(y)$ shows that $u(y)$ goes from $0\to-\infty$. This point is in the upper left plane and call it l_4.

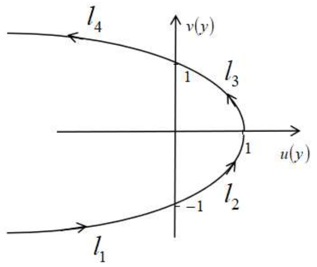

Figure 1. Trend diagram of point z

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As can be seen from the attached figure, when y goes from $-\infty \to +\infty$, $\eta = P(iy)$ rotates $\frac{1}{2}$ times counterclockwise around the origin. According to lemma $\underset{y(-\infty \to +\infty)}{\varDelta \arg} P(iy) = 2 \cdot \frac{1}{2} \cdot \pi = \pi$, $P(z)$ has three zeros in the left half plane $\operatorname{Re} z < 0$. End of proof.

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